FuE-Abschlussbericht

Numerische Simulation von hydraulisch induziertem Feststofftransport im Übergangsbereich zwischen Boden und Wasser

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Der Bericht darf nur ungekürzt vervielfältigt werden. Die Vervielfältigung und eine Veröffentlichung bedürfen der schriftlichen Genehmigung der BAW.
Zusammenfassung

Bed protection measures on federal waterways consist of armor stones which rest on top of filter layers or geosynthetics. These are put in place to protect the subsoil from scour and erosion due to flow forces and also forces occurring due to the movement of ships. Rapid changes in water levels due to passage of ships, termed “drawdown”, relieves the mechanical load off the bed of the waterway and simultaneously establishes a very strong gradient in a manner that the pore-water pressure is at a higher value than the pressure of the water column above it. The current research is the first step to investigate stability problems arising in such situations on federal waterways.

Most of current approaches to simulating such phenomena have been grid-based, where an explicit connectivity is required between two grid points in order to calculate the values of the variables and derivatives between them. These methods face limitations when there are large deformations such as when the stability of the bed of the waterway is compromised and the material around the armor stones is washed out. When trying to simulate this in a grid based method, the simulation is only reliable under moderate distortion of the grid. For such severe deformations, a class of numerical methods known as meshfree methods is utilized. One among such methods is Smoothed Particle Hydrodynamics. In this method values of variables and derivatives are estimated as a weighted sum of the values of the neighbors which can move freely in a Lagrangian framework.

Previous work done in several fields such as heat conduction, linear elasticity, diffusion and computer graphics are utilized to reproduce the physics of the saturated groundwater system in SPH. It has been used to solve a pure diffusion problem in a rigid medium to emulate diffusion of pore pressure in an isotropic homogenous medium. Starting from this, the concepts of linear elasticity which have already been well established in SPH may be combined with the volumetric effects caused by the changes in the pore-water pressure to obtain a poroelasticity model. Here a comparison to thermoelasticity serves as a rough guideline for modeling. The final aim of the research is foreseen to be an implementation of a failure criterion and a flow rule so that the bed of the waterway can be simulated well beyond its failure point to investigate the post failure state of the system for various boundary and initial conditions.

In this report the fundamentals of modeling such phenomena from the point of view of mathematical modelling and from the point of view of using the mathematical model in the meshfree numerical method SPH is elaborated and basic examples of implementations are discussed.
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Chapter 1

Introduction

Waterways are complex systems which are composed of several interacting natural and artificial components. The natural components are the flowing water, the banks, the bed of the waterway, the groundwater in the soil, the flora and fauna present. The artificial components of waterways include hydraulic structures, the artificial bank protection measures in place and the ships which use the waterway.

One of the factors that play a role in the life of waterways is the bank and bed protection. On federal waterways the bed protection is made up of armor stones which act as a protective layer protecting the underlying ground from scour and erosion. Due to the highly interacting nature of the system, the flowing water in the waterway and the water in the groundwater system are connected to each other and exchange takes place between the two. This exchange not only influences the hydraulic parameters of the groundwater system, but also the mechanical properties of the ground and under certain highly dynamic conditions, can negatively affect the stability of the bed. The study of the processes in the system described above is crucial to identify circumstances under which the armor stones lose their stability and to infer how the system behaves after the stability has been affected or lost.

Numerical methods contribute greatly in the study of such highly complex and dynamic systems. Generally, in the field of geotechnical engineering, the numerical methods which are widely in use to assess the stability of soils are grid based methods and in particular the finite element method. However, the finite element method or the grid based methods in general face severe difficulties in the form of grid distortions which decrease the computational accuracy or fail (Bui et al., 2008) when it comes to handling huge deformations such as the ones that occur in hydraulic heave of the bed of waterways or embankment failure. To overcome the difficulty associated with mesh based methods, the usage of mesh-free methods has been steadily increasing in recent years. Most of the mesh-free methods, however, are used primarily to model highly violent phenomena such as astrophysical flows, free surface flows, wave interactions...
etc. In this research project, an effort has been made to conceptualize a model to be implemented in the mesh-free numerical method Smoothed Particle Hydrodynamics (SPH) and the implementation of a basic groundwater flow model has been described in detail.

1.1 Background

The section of earthworks and bank protection of the Bundesanstalt für Wasserbau (BAW) deals with conventional as well as alternative methods of bank protection. Conventional methods comprise of armor stones with either filter layers or geosynthetics while alternative methods such as using vegetation to stabilize banks are also being tested and evaluated. With conventional filters and geosynthetics, the aim is to provide a base for the armor stones, which are stable against erosion in changing hydraulic situations. The stability of the filter is important to maintain the armor stones at their intended place so that their functional efficiency is maintained. An example can be found in figures 1.1 and 1.2.

Figure 1.1: Armor stones for bank protection measures on waterways with a conventional filter underneath (BAW (2013a)).

The aim of these protection measures is not only to protect the bed from the hydraulic forces occurring during ship movement such as breaking waves and turbulence, but also to protect them from failure in the case of excess pore water pressure occurring during rapid changes of the free water surface. Based on the type of the underlying soil, one of three types of armoring is used. They are armor stones without grouting (fig. 1.3), partially grouted (fig. 1.4), and fully grouted armor stones (fig. 1.5).
Figure 1.2: Closeup of a bank showing, armor stones and a two-layered grain filter (BAW (2013a)).

Figure 1.3: Armor stones for bank protection measures on waterways with a conventional filter underneath (BAW (2008)).

The criteria which determine the stability of protection measures in waterways are the hydraulic conditions and the interaction between surface water and groundwater. The former influenced by passage of ships, the weather, seasonal and meteorological conditions. During times without traffic, the water level in a waterway is fairly constant, and the pressure exerted by this depth of water on the ground and the groundwater system would have reach equilibrium. Along such a section of a waterway when a ship passes by, there is a corresponding drop of water level to accommodate for the constriction. The sudden drop in water level which is termed drawdown (see fig.1.6) creates a condition where the groundwater and the ground are suddenly relieved of the weight of water. This creates dynamic conditions in the soil and in the groundwater (see fig.1.7). For the soil there is a temporary reduction in the weight of the water column above it. For the groundwater system, there is a temporary increase in gradient causing greater flow into the waterway since the groundwater is now at a higher pressure than the free water in the waterway. This system will have to equilibrate with the flow of a corresponding amount of water out from the bed and banks into the waterway or, until the water level has returned to its original state so that equilibrium conditions can again be established.

These processes are however on very different time scales. The time taken for the water
to return to its original level is far lesser than the time it takes for the excess pressure in the groundwater system to dissipate. The reason being the permeability of the ground, which is not high enough to allow the pressure differences to equilibrate fast enough with the changes in the surface water. This phenomenon is however only possible when the constituents of the groundwater system together possess a certain compressibility. Completely rigid and incompressible constituents show the exact opposite behavior since pressure changes are propagated at the speed of sound throughout the system.

Considering compressible constituents for now, the load decrease on the ground is of immediate effect. But, while the water level recuperates in the waterway, the load on the soil is much lesser compared to the pressure of water in the groundwater system and this excess pressure effectively acts outwards from the soil resulting in a decreased stability of the soil. This can lead to loss of self weight in the filter layer causing the armor stones to sink into the soil. Owing to the complex nature of this system, numerical methods are utilized to recreate the mentioned phenomenon and to study it in detail.

In the longer term, this study, along with subsequent studies is aimed to help the formulation of the guidelines and technical codes of practice for federal waterways. Currently the following codes of practice (in German Merkblatt) are seen to benefit in the longer run.

- Merkblatt Anwendung von Kornfiltern an Bundeswasserstraßen (MAK)
- Merkblatt Anwendung von Regelbauweisen für Böschungs- und Sohlsicherungen an Binnenwasserstraßen (MAR)
- Merkblatt Grundlagen zur Bemessung von Böschungs- und Sohlsicherungen an Binnenwasserstraßen (GBB)
1.2 Statement of the problem

From the physical phenomenon described previously an engineering problem has to be formulated. Here, importance is first given to the stability of the ground underneath the armor stones. To model this particular problem, a section of the ground is taken to be a saturated groundwater system with a water column standing above it. The interface between the above two systems is of great importance since any influence from one system into the other should pass through the interface and it is also where armor stones are situated. In the following study a model is conceptualized for the coupled free-flowing water and groundwater system from the point of view of the safety of the ground and the armor stones. The engineering problem is to analyze the influence of periodically changing mechanical and hydraulic boundary conditions of the groundwater system from the point of view of the mechanical stability of the ground. The report
is structured in the following fashion

- The concepts and definitions required for modeling has been elaborated.
- Numerical methods are introduced and the working of the chosen numerical method is explained
- Previous work done in numerical methods on similar lines to the problem of interest is elaborated
- Few open source software in the field of smoothed particle hydrodynamics are listed and briefly explained
- The implementation of a simple model describing a diffusion equation is explained. Suggestions for implementation of other models is given
- Possible methods to implement a coupled system are explained along with hints about directions in which further research may be pursued.
Chapter 2

The Model Concept

Recognizing the physical system is the first step of the modeling process. Next the physical systems have to be conceptualized a combination of simplified sub-models which capture specific processes which are of interest for the ease of modeling. Here, the system (fig. 2.1) can be divided into the following sub-models

- The waterway system which has free flowing water governed by the hydraulic conditions of the waterway and also influenced by the movement of ships.

- The saturated\(^a\) groundwater system of the bed of the waterway which has a soil component and a water component. The flow of water in the soil is governed by the flow-parameters of the soil (explained in sec 2.1, their changes and also the exchange this system has with the free flow. The soil itself possesses mechanical properties (explained in sec. 2.2) is governed by the mechanical load and by the changes in the groundwater flow e.g. pressure and flow forces.

- The region of exchange where these two models interact and influence each other. This is the region where water is exchanged from the free flow region to the ground-water system and the forces are transferred. The interface is also the region which is important from the point of view of stability since it is in this region that the armor stones are placed to protect the bed.

Although a bidirectionally coupled system is conceptualized, preliminary focus has been on the influence of sudden changes in water level on the groundwater system and consequently on soil stability. This is because, in the other direction, the changes in the groundwater system do not affect the major flow patterns of the waterway and based on this assumption the coupling in the direction groundwater to surface water has not been considered in this work. This is a simplification for modeling purposes

\(^a\)Here the term saturated indicates that the pores are completely filled with one liquid phase, here water. The water might have components dissolved in it (e.g. gas) which alter its mechanical properties or the pore spaces might have air bubbles trapped in them in residual saturation, but for the purposes of modeling only one liquid phase is considered to be filling the pore spaces.
Figure 2.1: Simplified model showing the flow in a section of a waterway, the turbulence between the armor stones and the exchange between then groundwater and the waterway

based also on the fact that a study of the macroscopic flow properties of the waterway is not the focus of this research. This would not be the case if there were to be any material transport from the groundwater into the waterway.

In the subsequent sections, this model concept has been converted into a mathematical model by listing out the processes occurring in each of the sub-domain and the governing equations for those relevant processes. Due to the inherent complexity of the system, this process has been done step wise by breaking the system down into two individual sub-models. The free flow system was preliminarily investigated and readily implemented free flow models were modified to suit the problem at hand and simulated using the SPH code. But, due to lack of appropriate boundary conditions for the free-flow region in the simulation software and also since from the application point of view, the most important processes of interest occur at the interface between soil and water, the investigation of the free-flow region was no longer pursued. Instead a the focus was laid on implementing the physics of the flow in soils and the resulting mechanical response of soil under dynamic conditions.

To describe the soil model mathematically, it is necessary to introduce certain physical parameters, their definitions and the influence they have on the system. These have been listed and defined in the next sections.
2.1 Flow properties of soils

2.1.1 Porous Medium and Representative Elementary Volume

The term “porous media” deals mainly with not-completely-solid objects under the additional consideration that they allow a non-solid phase to exist and flow through them. This requires that there not only are pore spaces in porous media but also that many of these pores are connected to each other to allow fluid flow. For an elaboration on the definition of porous media with additional literature, refer Bear (1988, c1972) and the citation mentioned therein.

To model processes in such a multiphase system, there can be two approaches. In the first approach the geometry of the pore spaces is measured and reconstructed. The flow is then modeled by the equations of fluid flow solved individually in each pore space with boundary conditions imposed to the adjacent soil grains in each pore. Owing to the complexity of determining pore geometries in all but the most simple real-life cases, a second approach is taken namely the Representative Elementary Volume approach as mentioned in Bear (1988, c1972). In this approach, the porous media is considered at a spatial scale which is much greater than the pore scale, but much smaller than the entire domain such that heterogeneities can still be resolved. At this scale the clear distinction between phases is no longer recognizable and flow need not be modeled in each pore space. Instead, the volumetric composition of phases within the Representative Elementary Volume (REV) is found out and only the bulk volume changes of the solid or the fluid is modeled with new equations.

2.1.2 Porosity

The individual spaces between the pores when considered at the REV scale denote the hollowness of the REV. The porosity describes the part of the total volume of the REV of a soil available to be occupied by a fluid. The porosity for example plays a particular role in this study since it gives the percentage volume of the REV available for occupation by water at any given time. A change in porosity therefore implies a change in fluid content assuming the system to be fully saturated.

2.1.3 Intrinsic permeability and hydraulic conductivity

The intrinsic permeability is a measure of the ease with which any arbitrary fluid can flow through the considered medium. It is an absolute quantity with the units of $m^2$. Soils with high permeability can have a higher discharge at the same fluid pressure gradient acting across them as compared to soils with low permeability. It is a measure of the resistance offered to flow inside the pore spaces.
The intrinsic permeability of a soil is a property specific only to the soil. The hydraulic conductivity on the other hand, is a lumped property of the soil influencing the flow of a specific fluid. It is a property combining the intrinsic permeability of the soil and the properties of the fluid namely viscosity and fluid density. It gives the quantity of discharge of a particular fluid for a fluid pressure gradient for any particular soil and for a constant discharge per cross-section, it is inversely proportional to the pressure gradient. Therefore a soil of with a certain value of intrinsic permeability will be highly conductive for a fluid of low viscosity and/or density, while being poorly conductive for a fluid of high viscosity and/or density. The hydraulic conductivity has the units \( m/s \). The relation between the hydraulic conductivity \( k \) and the intrinsic permeability \( K \) is

\[
K = k \frac{\mu}{\rho g}
\]

(2.1)

therefore the hydraulic conductivity is inversely proportional to the kinematic viscosity which is defined as

\[
\nu = \frac{\mu}{\rho}
\]

(2.2)

2.1.4 Storage

In soils holding water, storage is the measure of the quantity of water held in a unit volume of soil at any pressure. When the pressure increases even beyond the point when all the voids of the soil have been filled, then the elasticity of the soil body is utilized and some quantity of water goes into storage by compressing the soil skeleton. For purposes of simplicity it is here assumed that only the elasticity of the soil matrix contributes to the storage of the soil and not the compressibility of individual grains. Although the compressibility of pure water is relatively small, the inclusion of dissolved gas increases the the compressibility greatly (to the same order of magnitude of the compressibility of the soil matrix) (Langguth and Voigt, 1980; Cirpka, 2004). The storage of the soil matrix, in combination with the hydraulic conductivity determines the transient response of the groundwater system and the time taken to reach steady state. From the specific storage, a dimensionless storage coefficient can be derived by integrating over the depth of the soil.

2.1.5 Theory of Porous Media

The theory of porous media deals with the mathematical description of the flow phenomena which occur in soils and other porous structures at a continuum level. Here equations derived in continuum mechanics are used to solve (coupled) flow and mechanical problems in saturated soils at the REV scale. The first step towards such a description is to define the spatial scales at which porous media is to be considered.
Although many spatial scales exist, right from resolving individual grains up to very large scales covering meters are even more, the minimum volume at which the system can be described as multiphase continuum is the one where the system is considered large enough to avoid resolving individual pores while still being able to distinguish heterogeneities. At this scale the equations of the theory of porous media are formulated. This volume is denoted as the representative elementary volume of the system. The system is considered at this new macroscopic level and all the quantities which influence the flow at the microscopic (pore) level are averaged over this volume giving rise to new parameters and equations (Bear, 1988, c1972).

### 2.1.6 Flow in Porous Media

Darcy’s law gives an expression of the velocity of the fluid relative to the porous medium. It is a linear proportionality law that is valid for purely viscous flows in porous media with Reynolds number less than 1\(^b\) and relates the velocity of the fluid to the gradient of hydrostatic head the porous medium. The constant of proportionality is the hydraulic conductivity.

The above description for the flow velocity in porous media is for the case when the gradient of the hydrostatic head does not vary with time. Under dynamic conditions there are extra terms which influence how the system behaves. In a system at steady state, when the pressure of the water is varied at one region of the domain, a local gradient is created which forces the water from the region of high hydrostatic head to the adjacent pores. This happens until the entire system adjusts to the new pressure and a new pressure gradient is set up. Therefore this process of flow of water from one pore to another takes a finite time, and on the macroscopic scale, this can be seen as a propagation of the hydrostatic head through the system. This propagation is determined by the properties of the soil namely the storage and the conductivity.

Here an analogy can be drawn to heat transport in a solid through conduction. In this case when there is a change in temperature at one end of the domain, steady state is said to be reached when the change has propagated and the entire domain has adjusted to the new temperature gradient. The time taken for this propagation is calculated based on the length of the domain, the specific heat capacity and the thermal conductivity. For example if the specific heat capacity of a system is very high then it is evident that the change takes longer to propagate. If the conductivity is high then the changes are propagated much faster. The ration of the specific heat capacity and the conductivity is called thermal diffusivity. This is now extended to groundwater flow under dynamic conditions.

\(^{b}\)In the REV scale, the characteristic length used to determine the Reynolds number is usually the mean grain diameter (here \(d_{10}\)) or the mean pore dimension. The characteristic velocity is the specific discharge for which the Reynolds number is to be determined.
Assuming the volumetric water content to be equal to the porosity of the medium, any change in the porosity or in the density of water is due to the a non-divergent flow field

\[
\frac{\partial(\phi \rho_w)}{\partial t} + \nabla \cdot (q \rho_w) = 0
\]

(2.3)

where \( q \) is specific discharge given by the Darcy’s law

\[
q = -k \nabla h
\]

(2.4)

substituting, expanding, neglecting the spatial variation of the density of water and writing the time derivatives as a chain rule with respect to the hydrostatic head equation 2.3 becomes

\[
\left( \frac{\partial \phi}{\partial h} + \frac{\phi}{\rho_w} \frac{\partial \rho_w}{\partial h} \right) \frac{\partial h}{\partial t} - \nabla \cdot (k \nabla h) = 0
\]

(2.5)

In the groundwater system, this storage gives a measure of how much water is stored in or released from the pore space for any change in pressure (Cirpka, 2004). It combines the change in porosity and density of water for a change in head.

It can be derived by substituting Darcy’s law into the law of conservation of mass in porous media. It is also clear that the storage characteristics are strongly dependent on the compressibility of the soil skeleton and that the soil responds volumetrically with the increase of pressure. The soil skeleton compresses and results in an increase of pore volume which can accommodate more water. This at the macroscopic scale is observed as water “going into” and being “released from” storage. From eq. 2.5 it can be observed that the propagation of pressure changes is controlled by the parameters of the equation namely the storage term and hydraulic conductivity. This has the form of a diffusion equation with hydrostatic head as the diffused quantity. This equation can also be rewritten to use pressure pressure instead of using the hydrostatic head, but the equations in further sections (e.g., linear elasticity) utilize the excess-pore-pressure formulation of these equations as they can be readily related to available literature.

### 2.2 Mechanical properties of soils

#### 2.2.1 Stresses and Strains

In soils there can be three different types of stresses resulting from three different types of forces that are usually applied namely

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\(^4\)Here and in the rest of the chapter, the variable \( h \) denotes the hydrostatic head which is defined as the length of a imaginary water column which is equivalent to the pressure at the point being measure. In later chapter \( h \) also stands for smoothing length of the SPH notation.
• Compressive forces
• Tensile forces
• Shear forces

In engineering applications, it is desired that a soil takes compressive loads. However, there are situations where a soil may be subjected to shear loads and in some other cases where an interaction with groundwater needs to be considered, the tensile strength of the soil also becomes important. Since a soil is made up mostly of grains of minerals and voids between them, any volumetric behavior may be assumed to be arising due to the change in these voids and the rearrangement of the grains of the soil. This assumption is made since the compressibility of minerals due to engineering loads is very small and therefore can be neglected without greatly affecting the accuracy of the model (Muir Wood, 2004). From this it can be inferred that the changes in normal loads to a soil cause changes in volume of the soil. This change in volume as a ratio to the original volume is called volumetric strain.

Apart from the normal forces on surfaces, there may also be tangential forces which develop shear stresses in the soil and the soil responds by rearranging the grain structure appropriately to the stresses developed. These shear stresses, therefore, give rise to shear or deviatoric strains. Within certain limits of strain, these shear strains can be taken to involve only a change in shape of the soil while the volume can be assumed to remain unchanged. However, there are certain soils which rearrange their grains in such a way that the volume of pores available for occupation by a fluid may increase or decrease. These soils are called respectively contractant and dialatent soils.

Mathematically, stress $\sigma$ is defined as a force $F$ acting on an area $A$.

$$\sigma = \frac{F}{A} \quad (2.6)$$

If this stress produces a change in length $\delta L$ in the body of length $L$ then the strain $\epsilon$ in the body is defined by

$$\epsilon = \frac{\delta L}{L} \quad (2.7)$$

For a three dimensional body, there can be three normal stresses and six tangential stresses\(^d\). These cause three normal or isotropic strain, and three tangential or deviatoric strains. The components of the stresses $\sigma$ and strains $\epsilon$ tensors can be written in Cartesian form as\(^e\)

\(^d\)Variables and operators written in bold are intended to be read as tensors and tensor operators.
The total stresses tensor can further be written as a sum of a pure (isotropic) stress tensor which has entries only along the principal diagonal and a pure shear (deviatoric) tensor which is traceless.

\[
\sigma_{ij} = \frac{1}{3} \delta_{ij} \sigma_{kk} + \left[ \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \right] \tag{2.10}
\]

\[
\epsilon_{ij} = \frac{1}{3} \delta_{ij} \epsilon_{kk} + \left[ \epsilon_{ij} - \frac{1}{3} \delta_{ij} \epsilon_{kk} \right] \tag{2.11}
\]

Stresses and strains are interrelated through the material constants. These proportionality constants in the material law relating the changes in stress to changes in strain.

In addition to the normal and tangential stresses, there is a third source of stress in the soil which is the fluid filling the pore spaces. The fluid, here usually water, has a hydrostatic pressure distribution which is linear with depth and develops an outward normal stress against the combined compressive stresses developed from the self-weight of the soil and from the weight imposed on it (overburden). This acts as a buoyant force and reduces the effect of the loads on the soil. This effectively reduces the stresses in the soil and the resulting stress is therefore termed “effective stress”. Furthermore the pressure of a fluid at one point may be influenced by events occurring much further away from the point under consideration since fluid in the pore spaces is a continuum. Hence, a change in the pressure can be transmitted from the point of origin radially outward in all directions. The speed of propagation, however, depends on the elasticity of the continua. Hence it is to be expected that in a highly compressible fluid or in a medium with relatively high storage (as compared to hydraulic conductivity), disturbances will not be propagated very far into the medium from the point of origin.

### 2.2.2 Stiffness of soil to loading

Strains in the soil affect the size and shape of the voids in the soil. The changes are due to compression or expansion of the soil skeleton in the normal direction and due to grains slipping past each other in the tangential direction. In saturated soils, the

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*Throughout this chapter the subscripts \( i, j \) and \( k \) in mathematical expressions refer to the indices of the tensors (cf. index notation). These are not to be confused with the same alphabets used for naming particles in subsequent chapters where a new convention will be defined.*
response to normal stresses depends greatly on the pore water. Since the compressibility of soil whose pores are filled completely with water is much lower than one whose pores are filled completely with air, it is expected that water-saturated soil can take more load or will be stiffer. When an external load is applied on a saturated soil the response of the soil depends on the hydraulic boundary conditions of the soil specimen, that is, whether the faces of the soil specimen are drained or undrained. If the water in the pores are allowed to drain quickly such as in highly permeable soils, then the load brought upon it is very quickly taken completely by the soil and the soil reacts to the load by reducing the pore spaces and “consolidating”. In contrast if the water is allowed to drain very slowly such as in low permeable soils, then the load is first taken entirely by the water, this forces the water out of the pores till the pressure equilibrates with the applied load. It is only when the water no longer drains out of the soil body that the load is completely taken by the soil and the soil “consolidates” in the same manner explained before (Kolymbas, 2011).

2.2.3 Linear Elasticity

The stresses and strain definitions have already been introduced in section 2.2.1, but it still has not been defined how changes in stresses cause changes in strains or the other way round. This response of a solid material can be described by the theory of linear elasticity. This theory relates the stresses in a solid body as a function of strains and the material constants or vice versa. Linear elasticity is a simplified model of elasticity with the following considerations Detournay and Cheng (1993)

- Small strains
- Changes in stress depends only on changes in strain
- Once the loading of the body is removed, the stresses are returned to their prior values

This is chosen at first, as subsequent model complexities can be built upon the model to include more complex behavior like a failure criterion and plastic deformation. Based on this the relation between stress and strains can be written as

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \] (2.12)

referring to 2.2.1, the stress (\(\sigma_{ij}\)) and the strain (\(\varepsilon_{kl}\)) tensors have 9 components each which makes the stiffness or the elasticity matrix (\(C_{ijkl}\)) to have 81 components, however, some symmetries can be used to reduce the components of the stress and strain tensor to 6 each and the elasticity tensor to 36. The following hold for the above mentioned case.

\[ \sigma_{ij} = \sigma_{ji} \] (2.13)
\[ \varepsilon_{ij} = \varepsilon_{ji} \] (2.14)
\[ C_{ijkl} = C_{jikl} = C_{ijlk} \] (2.15)
Figure 2.2: Schematic showing relation between force, stress, strain and displacement through the material law.

Many different sets of material parameters can be established to define the components of the elasticity tensor, but only any two parameters are needed to determine all other parameters uniquely. These are determined by experiments. Under the assumption that the system is isotropic, the equations for linear elasticity simplify to the formulation of the Hooke’s law in one of its many forms using the Lamé’s parameter and shear modulus

$$\sigma_{ij} = \lambda \text{tr}(\epsilon) I + 2G\epsilon \quad (2.16)$$

Further simplifications can be made for 2D conditions prescribing either the stress or the stain on any one plane. In plane strain conditions the strains in one plane (here $\epsilon_{zz}$) is set to zero so the corresponding stress components are

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = \sigma_{ij} \quad (2.17)$$

and likewise in plain stress the stresses on one plane (here $\sigma_{zz}$) are set to zero and the corresponding strain components are

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} = \epsilon_{kl} \quad (2.18)$$

Accordingly the elasticity tensor is greatly reduced and contains now just 5 components.

$$\epsilon_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \quad (2.19)$$
where \( u \) is the displacement. The displacement is used as a primary variable when the numerical method stores an original configuration and compares the movement in each time step of simulation to the original configuration. The difference between the present configuration and the original configuration is the displacement vector. In Lagrangian methods which are pursued in this research, the points of a body are followed along their movement and hence the original configuration of the body is not stored. In this approach the laws of linear elasticity will be converted to rate formulations to be implemented in the numerical method. This is explained in 3.5.1.

### 2.3 Properties of soil as a multiphase continuum

#### 2.3.1 Poroelasticity

In the theory of poroelasticity, the stress strain relations mentioned in 2.2.3 are modified to include a pore fluid and its effect on the porous media. The pressure of the pore fluid changes the volumetric response of the solid and in the opposite direction, the volumetric change of the pores of the solid changes the pressure of the fluid and hence the flow. The theory of linear elasticity as described in the section 2.2.3 should now be changed to include two more variables. The pressure of the fluid in the stress part and the corresponding change in the volumetric water content in the strain part.

Accordingly new constitutive relations can be derived which are explained in detail in Detournay and Cheng (1993). Although pores usually have air trapped along with the liquid, for reasons of simplicity the pore space can be assumed to be fully saturated with a non gaseous liquid, in this case water. Pure water being negligibly compressible, it can be assumed that a change in water content of the pores spaces is exactly equal to the change in volume of the pore space. This coupling of the volumetric strain of the soil to the change in mass content of the fluid in the pore space is the first step. The second step is to couple the fluid pressure and it’s changes to the stresses in the soil. This is done by the concept of effective stresses as has been described by Terzaghi. The effective stress tensor of the soil is the difference of the total stress and the pore pressure.

\[
\sigma' = \sigma - pI
\]  

(2.20)

where \( I \) is the identity matrix. The concept of Terzaghi’s effective stresses when applied to the linear elastic constitutive law dictates that the strains should also be as effective strains, “effective” in the sense that the total strain of the soil be deducted by the isotropic volumetric strain caused due to change in pore pressure. However in order to describe the changes in effective stresses in terms of the total strains, the contribution to the volumetric strain due to pore pressure changes was incorporated by Biot (1941). By considering two further elastic moduli, namely the compressibility of the soil skeleton and the compressibility of the soil grains, the effective stresses according to Biot is
defined by

\[ \sigma' = \sigma - \alpha p I \]  

(2.21)

where the Biot’s coefficient \( \alpha = 1 - \frac{K}{K_s} \), represents the ratio of the bulk moduli of the soil skeleton and the soil grains\(^1\). From this it is to be concluded that the effective stresses in a linear elastic constitutive law can be described in terms of total strains only if \( \alpha = 1 \) (implying the bulk modulus of the soil grains are much higher than the bulk modulus of the soil skeleton \((K_s \gg K)\)). In all other cases the effective stresses are to be given in terms of effective strains (Andersen). For modeling phenomena in soil, the latter case is irrelevant as grains of soil are made of minerals which are extremely rigid compared to the soil skeleton.

### 2.3.2 Equations of motion

The constitutive laws explained are now combine into the equations of motions for the solid and fluid body. The Cauchy momentum equitation gives the equation for motion of the solid body

\[ \nabla \cdot \sigma' + f_b = \rho \frac{\partial \mathbf{v}}{\partial t} \]  

(2.22)

where \( f_b \) are the set of body forces, usually gravity. The steady state form of this equation is extended in the classical literature by substituting the stress with the strains and the material constants using the material law. The strains are further substituted as the gradients of deformations giving rise to the Navier type equation. However, for the current purpose the above form of the Cauchy’s momentum equation is maintained and implemented in the numerical method. Due to the Lagrangian method of implementation, the partial temporal derivative, is reduced to an ordinary temporal derivative following the motion. The temporal changes in stress itself is calculated separately and integrated explicitly in time. This will be explained in section 3.5.

The dissipation of excess pore pressure is expressed by a diffusion like equation drawn from the analogy of heat transport. Here the transported quantity is the excess pore pressure, this is the pressure which is more than the hydrostatic pressure of the pore water for that particular pressure condition. This is given by

\[ \frac{\partial p}{\partial t} - K_M \nabla^2 p = 0 \]  

(2.23)

Where \( M \) is the inverse of the storage coefficient, termed the Biot’s modulus. Therefore by definition the Biot’s modulus gives the change in pressure due to a change in volumetric content of fluid in the pore space at constant volumetric strain. This change in the excess pore pressure also leads to the volumetric strain of the soil. Therefore

\(^{1}\)In the next chapter \( \alpha \) is also used as the coefficients for SPH kernel functions. Both are, however, unrelated
2.23 is linked to the rate of strain of the soil by combining the law of conservation of mass (with volumetric content of the soil $\zeta$)

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot q = 0 \quad (2.24)$$

and the Darcy’s law for $q^g$

$$q = -\frac{K}{\mu} \nabla (p - \rho g) \quad (2.25)$$

and the relation from Detournay and Cheng (1993)

$$p = M(\zeta - \alpha \epsilon) \quad (2.26)$$

as

$$\frac{\partial p}{\partial t} - KM \nabla^2 p = -\alpha M \dot{\epsilon}_{vol} - M (K \rho g) \quad (2.27)$$

for details, refer Detournay and Cheng (1993). This equation couples the volumetric response of the soil on the right hand side in terms of the Biot’s modulus and Biot’s coefficient as coefficients for the volumetric strain rate, which together with body forces influence the rate of dissipation of pore pressure, given by the terms on the left hand side.

### 2.3.3 Implications on Stability

While applying these equations to the problem of interest, the volumetric response of the material to loads has to be considered. There are two types of volumetric response of a saturated fluid as explained in sec. 2.2.2. For the problem of rapid drawdown in waterways, a decision is to be made as to whether the material should be considered drained or undrained during the drawdown and the subsequent recovery process. To aid the decision, the dynamics of the diffusion equation is made use of. The diffusion equation of groundwater flow without poroelastic effects, with the storage term and the conductivity integrated over the depth of flow, without sources or sinks, in one dimension reads\(^b\)

$$\frac{\partial h}{\partial t} - T_{xx} \frac{\partial^2 h}{\partial x^2} = 0 \quad (2.28)$$

choosing a characteristic head difference and a characteristic length scale, the above equation is written in the non dimensional form as

$$\frac{\hat{h}}{\partial t} - \frac{T_{xx} \partial^2 \hat{h}}{SL^2 \partial \hat{x}^2} = 0 \quad (2.29)$$

\(^a\)Here $K$ is the intrinsic permeability tensor, and $\mu$ the viscosity. The intrinsic permeability tensor is used in place of the hydraulic conductivity since it is independent of the fluid flowing through the soil and hence provides a more general form of Darcy’s law.

\(^b\)The transmissivity in the x direction $T_{xx}$ is different from the temperature (in the next chapter) which also uses the variable $T$. For legibility, transmissivity is always written with subscripts since it is a tensor quantity.
since all variables now lack dimensions the ratio
\[
\frac{T_{xx}}{SL^2}
\]
has the units \( \frac{1}{t} \) and is called the diffusivity. Hence
\[
\frac{SL^2}{T_{xx}} = t_c
\]
represents a time. The time for a system of constant thickness and constant length is determined by the ratio of the storage term to the conductivity term. This represents the time in which the system will attain steady-state after a change in one of the boundary conditions. Soils with higher conductivity will attain steady-state quicker since \( t \) is inversely proportional to \( T_{xx} \). As an example for unit dimensions, a conductivity value of the underlying ground in the order of \( 10^{-4} \) (BAW, 2013b), MMB to conductivity values of filter layers above cohesive soils to the order of \( 10^{-5} \) ((BAW, 2013a), MAK) corresponds to a time value of the order of \( 10^4 \) to \( 10^5 \) seconds which implies a steady-state time of several days. But the drawdown and recovery phase corresponding to the movement of ships across any given cross section has a time scale of the order of maximum \( 10^2 \) seconds. This implies that the bed of the waterways or the filter layers do not have enough time to dissipate excess pore pressure cause by change in the hydraulic conditions due to ships movement. Furthermore, due to the contrast of the time scales by several orders of magnitude, the soil underneath the bed and the filter layers can be assumed to be subjected to undrained loads, or rather, undrained reduction of loads at the water soil interface. This sudden reduction along with the long time taken for pore pressure equilibration means that the filter layers and the soil below are under extreme pore water pressures for short periods of time, which drastically reduces effective stresses in those layers. This coupled with the flow and the inherent turbulence might give rise to shallow erosion problems, failure or destabilizing the armor stones placed on top of the filter layer.
Figure 2.3: Schematic of wave induced destabilization in low conductive soil (Timea Balint et al.).
Chapter 3

Smoothed Particle Hydrodynamics

3.1 Introduction

One of the many classifications of numerical methods is based on how they handle conservation laws, according to this there are two categories

- Eulerian methods
- Lagrangian Methods

Eulerian methods consider all the conservation laws from a fixed frame of reference in space and balances are drawn between the inflow, outflow and the local storage which have to conserve all properties. The Lagrangian methods on the other hand follow the motion along the underlying velocity and always do local balances while following the motion. This way the partial differential equations are converted to ordinary differential equations along the path of motion and advection terms do not have to be modeled explicitly. Under the Eulerian methods are the Finite Volume method, some types of finite element methods and the method of finite differences. Under Lagrangian methods came the Lagrangian finite element method, Discrete Element Method (DEM), Particle Finite Element Method (PFEM) and the mesh-free method SPH.

3.2 Smoothed Particle Hydrodynamics

SPH is a class of mesh-free methods which was originally developed for astrophysical applications by Gingold and Monaghan (1977) but has been extended and developed for solving problems in fluid mechanics by Issa (2005), to free surface flows by Monaghan (1994), to solid mechanics by Gray et al. (2001), conduction problems by Brookshaw (1994); Cleary and Monaghan (1999); Blowey and Craig (2005), fluid structure interaction by Shao (2010); Lenaerts (2009); Ulrich (2013); Bui et al. (2007); Bui and Fukagawa (2013).
Contrary to grid based methods, mesh-free particle methods discretized a continuum into a set of particles which are not and need not be connected to each other. This allows for motion of these particles far beyond the movement allowed by mesh based methods which are limited by the problem of mesh tangling. The possibility of movement of the particles also allows one to choose a Lagrangian frame of reference and track the particles’ movement along their paths. Once the continuum has been discretized into set of particles values of variables are estimated as a weighted sum over the values of the variable at the neighboring points. The weighting is determined by a function which has its highest value at the point of interpolation and decreases smoothly to zero at fixed distance from the center. This distance within which the values at neighboring particles are considered for the weighted sum is called the smoothing length, denoted by $h$. This ensures that only the neighboring points are included in the weighted sum and not the points farther away which have a lesser influence. This property of the weighting function (decreasing value to zero at a finite distance away from the particle) is called “compact support”. To derive it mathematically, first a function is represented in the integral notation

$$ A(r) = \int A(r') \delta(r - r') \, dr' $$  \hspace{1cm} (3.1)

where $\delta(r - r')$ is the Dirac delta function with

$$ \int \delta(r - r') \, dr' = 1 $$  \hspace{1cm} (3.2)

*The notation for the smoothing length $h$ is not to be confused with the notation for hydrostatic head (also) $h$ appearing in equations in the previous chapter.*
the Dirac delta function is replaced by a weighting function $W$ which, in the limit, reproduces the Dirac delta function. So the following two conditions hold for a weighting function $W$ just as they hold for the Dirac delta function $\delta$

$$\lim_{r \to 0} W(r, h) = \delta(r)$$

$$\int W(r) \, dr' = 1$$

Additionally, the compact support condition

$$W(r, h) = 0 \quad \forall \, r > \kappa h$$

Replacing the Dirac delta function with the weighting function the integral approximation is rewritten as

$$A(r) = \int A(r') W(r - r', h) \, dr'$$

The integral equation above can now be changed into a sum over the neighboring particles. Therefore, the value at the function $A_j$ at the particle $j$ can be calculated by the weighted sum of $A_i$ where $i$ is the number of neighbors. The variable of integration $dr'$ is changed by multiplying

$$\frac{\rho(r')}{\rho(r')}$$

then the product $\rho(r')dr'$ is the mass $m$ of the particle. The discretized equation reads

$$A(r) = \sum_j m_j A(r') W(r - r', h)$$

Just as the particle approximation for a function is derived above, the particle approximation for the first and second derivatives of the function are also straightforward derivations involving using the product rule to transfer the gradient from the function to be approximated to the weighting function. Since the weighting function can be precisely defined, the derivatives can also be calculated with good accuracy and hence only the value of the function at a point and the value of the weighting function and its derivatives is required for the approximation. For details and derivations, refer Price (2004); Monaghan (1992) or Liu and Liu (2003). Examples are given below with velocity as the discretized variable since this discretization is most commonly used to interpolate velocities.

\[\text{In this chapter subscripts } i \text{ and } j \text{ denote SPH particles. The indices for tensorial notation are changed to the Greek alphabets } \alpha, \beta \text{ and } \gamma \text{ and are written as superscripts in all SPH summation equations.}\]
\[ \nabla v = \sum_{j=1}^{N} \frac{m_j v_j \nabla W(r_i - r_j, h)}{\rho_j} \]  

This form has the drawback that the gradient of velocity is non-zero even if the velocity of the particles \(i\) and \(j\) are the same. To solve this, the approximation equation is reformulated with the density inside the gradient operator (using the product rule) as given below

\[ \nabla v = \nabla \left( v \rho \right) - v \nabla \rho \]  

and then transferring the gradient operator to the weighting function thus getting a formulation which ensures the gradient is zero if the value of the variable is the same at the particles \(i\) and \(j\)

\[ \nabla v = \sum_{j=1}^{N} \frac{m_j (v_i - v_j) \nabla W(r_i - r_j, h)}{\rho_j} \]  

similar formulations are also derived for the vector dot and cross products. The expression 3.10 works well with the estimation of velocity, however when the same expression is used for the calculation of pressure in a momentum conservation equation, although the pressure at two particles \(i\) and \(j\) are the same, there exists a force which is equal in magnitude and opposite in direction between the two. To capture this pairwise symmetric force for the conservation of momentum, the SPH formulation should be rewritten firstly by considering the product rule of the ratio \( \nabla \left( \frac{p}{\rho} \right) \) and then solving for \( \nabla p \)

\[ \frac{\nabla p}{\rho} = \rho \left[ \frac{p}{\rho^2} \nabla \rho + \nabla \left( \frac{p}{\rho} \right) \right] \]  

converting this expression to the particle approximation yields

\[ \nabla p = \sum_{j=1}^{N} m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W(r_i - r_j, h) \]  

which ensures pairwise symmetry of forces.

To calculate the second derivatives, there approaches have been found in literature. They are

- The direct approach used by Müller et al. (2005) by using the second derivative of the kernel function
- The double summation approach used by Jeong et al. (2003) using approximation of two first order derivatives
The combined first derivative and difference approach elaborated in Brookshaw (1985), Brookshaw (1994); Cleary and Monaghan (1999) and Cleary (1998). A comparison of the above three methods has been done by Fatehi et al. (2008). The first approach, using the second derivative of the kernel function has been shown to be sensitive to particle disorder, cf. Monaghan (1992). The second approach, using the double summation approach is limited in the sense that first derivatives need to be know at all interpolation points in order to formulate second derivatives from first derivatives. In a problem where the particles do not move relative to each other (e.g. thermal conduction, groundwater flow without deformation) the first approach may be used as there will be no particle disorder with time. The last approach is used in the current research since it avoids the problem caused with particle disorder and since it neither requires the knowledge of the first derivative of a variable at all interpolation points nor the knowledge of the value of second derivative of the kernel function. This is given by Brookshaw (1994) for heat diffusion equation with temperature $T$ and a diffusivity of $Q$ as

$$
\left( \frac{1}{\varrho} \nabla \cdot (Q \nabla T) \right)_i = \sum \nabla \cdot \left( m_j \frac{Q_i + Q_j}{\varrho_i \varrho_j} (T_i - T_j) \frac{(r_i - r_j) \cdot \nabla W_{ij}}{|r_i - r_j|^2} \right)
$$

Application of this discretization to the current research has been elaborated in sec. 3.7.

### 3.3 SPH Kernels

The quality of approximation of variables in SPH depends on the kernels chosen. The most obvious and simplest choice for a SPH kernel would be the Gaussian curve but since it does not have compact support, other piece-wise polynomial functions are chosen. These functions should possess smoothness and continuity at least to the order of the derivatives they have to approximate Liu (2010). Therefore when second derivatives are to be approximated from second derivatives of the kernels, the kernels should have at least $C^2$ continuity Liu and Liu (2010). Due to the choices made in the previous section regarding the choice of approximation for second derivatives, there no longer exists a requirement of calculating any derivatives of the kernel function higher than the order 1. One of the simplest and a very commonly used kernel in many SPH software is the cubic spline kernel.

$$
W = \alpha \begin{cases} 
1 - 1.5q^2 + 0.75q^3 & 0 \leq q \leq 1 \\
0.25(2 - q)^3 & 1 \leq q \leq 2 \\
0 & q \geq 2 
\end{cases}
$$

(3.14)
\[ \nabla W = \alpha \begin{cases} -3q + \frac{9}{4}q^2 & 0 \leq q \leq 1 \\ -0.75(2 - q)^2 & 1 \leq q \leq 2 \\ 0 & q \geq 2 \end{cases} \] (3.15)

Similarly, other common kernel functions are the quintic spline

\[ W = \alpha \begin{cases} (3 - q)^5 - 6(2 - q)^5 + 15(1 - q)^5 & 0 \leq q \leq 1 \\ (3 - q)^5 - 6(2 - q)^5 & 1 < q \leq 2 \\ (3 - q)^5 & 2 < q \geq 3 \\ 0 & q > 3 \end{cases} \] (3.16)

\[ \nabla W = -\alpha \begin{cases} (3 - q)^4 - 6(2 - q)^4 + 15(1 - q)^4 & 0 \leq q \leq 1 \\ (3 - q)^4 - 6(2 - q)^4 & 1 < q \leq 2 \\ (3 - q)^4 & 2 < q \geq 3 \\ 0 & q > 3 \end{cases} \] (3.17)

And the Wendland kernel

\[ W = \alpha \begin{cases} (1 - 0.5q)^4 (1 + 2q) & 0 \leq q \leq 2 \\ 0 & q > 2 \end{cases} \] (3.18)

\[ \nabla W = \alpha \begin{cases} 5q (1 - 0.2q)^3 & 0 \leq q \leq 2 \\ 0 & q > 2 \end{cases} \] (3.19)

Where the value of \( \alpha \) is dependent on the dimension of the problem. Refer Liu and Liu (2003) for further details regarding kernel functions.

### 3.4 Compressible SPH

Although the particle approximation can be applied to all variables of a system the density is not evaluated with the standard SPH interpolant but with the Lagrangian derivative of the SPH density interpolant. This gives the continuity equation

\[ \frac{d\rho}{dt} = -\rho (\nabla \cdot \mathbf{v}) \] (3.20)

Where \( \frac{d\rho}{dt} \) is the Lagrangian derivative. Its corresponding discretized form reads

\[ \frac{d\rho_i}{dt} = \sum_j m_j (\mathbf{v}_i - \mathbf{v}_j) \cdot \nabla_i W_{ij} \] (3.21)

This way the density of each particle at each time step is obtained and then integrated to get the temporal change. This change in density reflects in a change in pressure using the equation of state
\[ p = \frac{c_0^2 \varrho_0}{\gamma} \left[ \left( \frac{\varrho}{\varrho_0} \right)^\gamma - 1 \right] \]  

where \( c_0 \) is the reference speed of sound at the reference density \( \varrho_0 = 1000 \text{kg/m}^3 \) Cre spo et al. (2013). For a multiphase system like the groundwater system, this above mentioned concept of pressure as a function of density has to be further clarified. The density defined above applies to a single phase system where all SPH particles are either pure solids or pure fluids. In case of a saturated porous system where only a percentage of each particle is solid (based on the porosity) the density approximated in this sense will be the density of the multiphase continuum (since no phases can be explicitly resolved at any point in the continuum) and hence combines the densities of the water in the pore space and that of the individual soil grains (dry bulk density). Assuming weakly compressible constituents (Water and soil grains), which is done by taking the reference speed of sound high enough such that the change in densities are only within a small percentage of the reference density, the bulk density of the multiphase continuum will change only if the pore volume itself changes or the water content in the pore space changes due to change in the fluid pressure.

While describing the groundwater system, the pore-pressure (discussed in the poroelasticity section) is not a function of density of the pore water but follows its own (diffusive) evolution equation explained the in sec. 2.3.2. Therefore the pressure derived from the equation of state depicts an isotropic stress on the porous medium. Apart from the approach of solving for the density and using the equation of state to get the pressure, it can also be derived directly from solving the Poisson equation for pressure using matrix solvers. Details are mentioned in Cummins and Rudman (1999), Shao and Lo, E. Y. M. (2003) and for the specific case of porous media flow in Shao (2010). This approach is not pursued further and the weakly compressible approach is chosen for literature review.

### 3.5 Previous Work in SPH

In this section the previous work done in individual models which have been utilized to construct the deformable porous media model are explained.

#### 3.5.1 SPH for elasticity of solids

The main drawback of conventional SPH without any correction was the problem of tensile instability, which resulted in particle clumping when the stress states in the model went into tension. This deviates from reality since in any real material the atomic forces would allow it to handle tensile stress and still not fail till the plastic limit is reached (Monaghan (2000)). The detailed stability analysis was done by Swegle
et al. (1995) and a correction was proposed by Monaghan (2000) where he also ran comparative tests for deflection of beams and colliding rubber rings with and without the correction. This correction was then used by Gray et al. (2001) for modeling linear elastic materials which can withstand tensile stresses. In that example the rate form of the linear elastic material law was used as given below for the rate of change of deviatoric stress.

$$\frac{dS^{\alpha\beta}}{dt} = 2\mu \left( \dot{\epsilon}^{\alpha\beta} - \frac{1}{3} \delta^{\alpha\beta} \dot{\epsilon}^{\gamma\gamma} \right) + S^{\alpha\beta} \Omega^{\beta\gamma} + \Omega^{\alpha\gamma} S^{\gamma\beta}$$  \hspace{1cm} (3.23)

where $S$ is the deviatoric stress tensor given as a difference of the total stress tensor $\sigma_{\alpha\beta}$ and the isotropic stress tensor $\frac{1}{3} \sigma_{ii}$. Bui et al. (2008) used a similar formulation for the total stress rate to model large deformation and failure of geomaterials with an elasto-plastic constitutive law. In his paper the total strain was considered as a sum of elastic and plastic strains. The elastic strains given by

$$\frac{d\sigma^{\alpha\beta}}{dt} = 2\mu \left( \dot{\epsilon}^{\alpha\beta} - \frac{1}{3} \delta^{\alpha\beta} \dot{\epsilon}^{\gamma\gamma} \right) + \lambda \epsilon^{\gamma\gamma} \delta^{\alpha\beta} + S^{\alpha\beta} \Omega^{\beta\gamma} + \Omega^{\alpha\gamma} S^{\gamma\beta}$$  \hspace{1cm} (3.24)

These two equations are evaluated at each particle without drawing any relation to neighboring particles. Both Monaghan and Bui used the time derivative of the strain tensor $\epsilon_{\alpha\beta}$, the strain rate tensor $\dot{\epsilon}_{\alpha\beta}$ defined by the gradient of velocities as instead of displacements

$$\dot{\epsilon}^{\alpha\beta} = \frac{1}{2} \left( \nabla_\beta v^\alpha + \nabla_\alpha v^\beta \right)$$  \hspace{1cm} (3.25)

and $\Omega$ is the rotation tensor given by

$$\Omega^{\alpha\beta} = \frac{1}{2} \left( \nabla_\beta v^\alpha - \nabla_\alpha v^\beta \right)$$  \hspace{1cm} (3.26)

The above two equations in their discretized forms are written as

$$\dot{\epsilon}^{\alpha\beta}_i = 0.5 \left( \sum_j \frac{m_j}{\theta_j} (v_j^\alpha - v_i^\alpha) \nabla_j W_{ab} + \sum_j \frac{m_j}{\theta_j} (v_j^\beta - v_i^\beta) \nabla_i W_{ab} \right)$$  \hspace{1cm} (3.27)

and

$$\Omega^{\alpha\beta}_i = 0.5 \left( \sum_j \frac{m_j}{\theta_j} (v_j^\alpha - v_i^\alpha) \nabla_j W_{ab} - \sum_j \frac{m_j}{\theta_j} (v_j^\beta - v_i^\beta) \nabla_i W_{ab} \right)$$  \hspace{1cm} (3.28)

the Cacuhy momentum equation 2.22 was discretised with symmetric forces for the pressure term as mentioned in 3.12. Additionally, since dissipation is no inherent in SPH, dissipative terms were added in the form of the SPH artificial viscosity term $\Pi_{ij}$ according to Monaghan (1992).
\[ \Pi_{ij} = \begin{cases} -\alpha_{ij} \mu_{ij} + \beta \mu_{ij}^2 \frac{v_{ij} \cdot r_{ij}}{r_{ij}^2} & v_{ij} \cdot r_{ij} < 0 \\ 0 & v_{ij} \cdot r_{ij} > 0 \end{cases} \]  

(3.29)

and

\[ \mu_{ij} = \frac{hv_{ij} \cdot r_{ij}}{r_{ij}^2 + \eta^2} \]  

(3.30)

where \( \eta \) is a small constant to avoid singularities. Furthermore, in the same article an explanation is also given about the values to be chosen for \( \alpha \) and \( \beta \). Cleary (1996) introduced a viscosity approximation for flows based on physical viscosity as compared to artificial viscosity

\[ \left( \frac{1}{\theta} \nabla \cdot (\mu \nabla u) \right) = \sum_j m_j \frac{r_{ij} \cdot \nabla_i W_{ij}}{r_{ij}^2} \]  

(3.31)

which is the form that is also used for the calculation of second derivatives in subsequent sections. Apart from the viscosity correction, a correction to eliminate the tensile instability problem was suggested in Monaghan (2000) and the momentum equation was modified in Gray et al. (2001) and as

\[ \frac{dv_{\alpha}^a}{dt} = \sum_j m_j \left( \frac{\sigma_{\alpha\beta}^a}{\theta_i^2} + \frac{\sigma_{\alpha\beta}^j}{\theta_j^2} + \Pi_{ij} \delta_{\alpha\beta} + R_{ij}^{\alpha\beta} f^n \right) \]  

(3.32)

where factor \( R_{ij}^{\alpha\beta} \) is calculated by rotating the plane of the stresses through an angle \( \theta \) from its original orientation in order to make the stress tensor purely diagonal. To the diagonal elements of the rotated tensor, an artificial stress is calculated only to the diagonal stresses which are tensile. Therefore if \( \hat{R}_{xx}^i \) and is the correction term in the rotated coordinates and the x component of the rotated stress tensor of particle \( a \) is \( \hat{\sigma}_{xx}^a \), then

\[ \hat{R}_{xx}^i = -\epsilon \frac{\hat{\sigma}_{xx}^i}{\theta^2} \]  

(3.33)

where \( \hat{\sigma}_{xx}^a \) is calculated knowing the angle of rotation \( \theta \)

\[ \theta = \frac{1}{2} \tan^{-1} \left( \frac{2\sigma_{xyz}^i}{\sigma_{xx}^i - \sigma_{yy}^i} \right) \]  

(3.34)

and the relations given below relating the angle of rotation, and the stresses in the original and rotated coordinates

\[^a\alpha \text{ and } \beta \text{ here are constants for artificial viscosity formulation used in Monaghan (1992). The same naming conventions have been maintained so that the equations are readily comparable to the original literature.}\]
\[ \bar{\sigma}_{xx}^i = \sigma_{xx}^i \cos^2 \theta + 2 \sigma_{xy}^i \sin \theta \cos \theta + \sin^2 \theta \]  

\[ \bar{\sigma}_{yy}^i = \sigma_{xx}^i \sin^2 \theta + 2 \sigma_{xy}^i \sin \theta \cos \theta + \cos^2 \theta. \]  

Knowing \( \bar{R}_{xx}^i \), the artificial stress terms \( R_{xx}^i \) can be calculated for the original coordinates by

\[ R_{xx}^i = \cos^2 \theta \bar{R}_{xx}^i + \sin^2 \theta \bar{R}_{yy}^i, \]  

\[ R_{yy}^i = \cos^2 \theta \bar{R}_{yy}^i + \sin^2 \theta \bar{R}_{xx}^i \]  

and

\[ R_{xy}^i = \cos \theta \sin \theta \left( \bar{R}_{xx}^i - \bar{R}_{yy}^i \right) \]

This describes the behavior of a linear-elastic model when the strains are completely reversible within the elastic limit. At the elastic limit material failure occurs, and beyond the elastic limit the material is plastic. Bui et al. (2008) extended the linear-elastic model to include the plastic part by introducing an additional plastic strain tensor and the Drucker-Prager yield criterion. Post-Failure, he also elaborated about the associated and non-associated flow rules to describe post-failure flows but used the non-associated rule with a dilatancy angle of zero which implied that the material is incompressible in the plastic regime.

### 3.5.2 SPH for porous media flow

Flow through porous media has already been studied at the pore scale at many instances. The main idea behind it was explained by Morris et al. (1997) for modeling low Reynolds number flows using SPH. This was further extended by Zhu et al. (1999) for pore scale modeling of flow through porous media where simulations were performed at the pore scale solving the transient Stokes equation for creeping flow. In Zhu and Fox (2001) a diffusion equation was solved underlying the flow field to simulate mass transport at the pore scale and later in Zhu and Fox (2002) an advection diffusion equation was solved to simulate hydrodynamic dispersion. Additional to these, some applications of SPH to model porous-media flow at the macro scale can be found in Shao (2010) who used a Navier-Stokes like equation with two extension terms to govern the flows at two different regions. They are

- a linear term for low velocity (low Reynolds number) flow regimes and
- a quadratic term for high velocity (high Reynolds number) flow regimes

On a bigger spatial scale Lenaerts (2009) modelled porous flow for use in computer graphics and animation using a macroscopic approach. Here instead of solving the Navier-Stokes or the Stokes equation at all pore spaces, a simple mass diffusion model
was used based on Müller et al. (2005). In this model the diffusion of mass is formulated as a function of the Darcy velocity $v_j$ within a porous SPH particle (see fig.3.2)

$$\frac{dm_i}{dt} = \sum_j d_{ij} V_j m_j \nabla^2 W (r_{ij}, h_j)$$

(3.40)

where $d_{ij}$ is the coefficient determining the strength of diffusion formulated as a function of the Darcy velocity

$$d_{ij} = v_j \cdot \frac{r_j - r_i}{\|r_j - r_i\|} S^l_j$$

(3.41)

where $S^l$ denotes the saturation of phase $l$ (here liquid). The above equation is modified to derive an expression for $d_{ij}$ for a system with a single fluid phase by setting the saturation of all particles to one, which implies all pore spaces are fully filled with one fluid. Equation 3.40 is then integrated explicitly in time to obtain the mass of each particle at each time step.

Figure 3.2: Description of a continuum discretized by porous SPH particles (Lenaerts (2009)).

This approach is well suited to model phenomena occurring at large spatial scales. This approach, however, has no mention of any effects of flow on the porous skeleton. To include the effects of poroelasticity, an approach has been proposed here for modeling groundwater flow by drawing an analogy to heat transport in solids. The immediate analogies which can be drawn between are

- both are diffusive processes, one is a diffusion of pore pressure and the other is a diffusion of temperature.
- both are also governed by second order differential equations with an elliptic and parabolic part.
• the hydraulic conductivity of the soil has similar influence on the flow of mass as does the thermal conductivity on the flow of heat.

• the storage term has similar effect in the storage of mass as does the heat capacity on the storage of heat.

The influence of the flow on the soil (poroelasticity) has already been compared to the effects caused by the flow of heat in solids (thermoelasticity) Wang (2000), Norris (1992), Ling et al. (2009)(chapter 5).

### 3.5.3 SPH for Coupling Models

In the work from Lenaerts (2009) an attempt has been made to couple the fluid flow model and the porous media model. As explained in the previous section, the flow of fluid is considered as a diffusive process with the Darcy velocity and hence the pressure determining the strength and direction of diffusion. This pressure is linked to the effective stress principle 2.21 to determine the effective stresses of the solid. A description has also been given regarding the absorption phenomenon on the surface of the porous body by modeling the fluid particles on the surface as porous particles with porosity as unity. This is then absorbed by the porous particle of the body and the mass of the fluid particle on the surface of the porous body decreases as the fluid mass is diffused into the porous body. When all the mass has been diffused, then the fluid particle is deleted from the particle system. Mass conservation is explicitly controlled based on the mass of the fluid diffused and the volumes of pores available for the fluid (see fig.3.3).

![Figure 3.3: Schematic of coupling concept (Lenaerts (2009)).](image-url)
3.6 SPH Software

During implementation many different SPH software were first studied to evaluate which would be well suited to modification to fit to the problem at hand. All the codes chosen were open source and all are written in C++ with the exception of one which is written in C. The codes of the following software were analyzed

- DualSPHysics, with both GPU and CPU implementation of SPH based on SPHysics which was written in Fortran from Gómez-Gesteira et al. (2012)

- GPUSPH, a purely GPU based SPH solver written in CUDA and C++ from Héralt et al. (2010)

apart from the ones mentioned above, code reviews were also done for the ISPH software from Shao and Lo, E. Y. M. (2003), but was abandoned due to lack of proper documentation. GPUSPH was not pursued further owing to the complexity of CUDA programming which was not the focus of the current research. GADGET from Springel (2005), a N-body/SPH code for cosmological applications has been modified and extended by Ulrich (2013) as GADGET-H2O for simulating fluid structure interaction problems occurring in ports, however, it is not open source and hence has not been pursued further. DualSPHysics was chosen as the suitable software for preliminary implementation of the model.

3.6.1 DualSPHysics

The origin of DualSPHysics was from SPHysics which was rewritten from Fortran to run on multi-core CPUs and GPU. But the main functionality of SPHysics was still to simulate free-surface flows using SPH. The core SPH solvers of both SPHysics and DualSPHysics are open source, but certain pre- and post-processing parts are released only as binaries. The software has certain common files which handle both CPU and GPU parts of the code and then the control is transferred, based on whether a CPU or a GPU executable is compiled, to the respective exclusive sections. In each of the section, a separate SPH solver is implemented.

The documentation of DualSPHysics is available along with the code and examples on the DualSPHysics website. To highlight it, it has the following options

- Kernel Functions
  - Cubic spline
  - Wendland Kernel
- Time integration schemes
  - Verlet
– Symplectic

• Viscosity implementation
  – Artificial viscosity
  – Laminar viscosity with Sub particle scale turbulence

The program uses an XML file with the suffix “_Def” to store information about

• the simulation constants such as the value and direction of gravity, values of
  parameters for the equation of state, the reference density, the CFL number for
  the time stepping.

• the geometry information such as particles per point, spacing of particles, position
  and size of boundaries, solid objects, particle identifier values

• the simulation parameters such as the time step, stepping scheme, type of kernel,
  viscosity implementation, density filters and the total simulation time.

This is then fed to the the program binary gencase, which generates another XML with
additional parameters to be used by the mail SPH solver. These additional parameters include

• the number of particles in the domain, generated by gencase

• the smoothing length

• mass of the boundary and fluid particles

The program is implemented with the following flow

• The method “LoadConfig(cgf)” loads the configuration (geometry, parameters,
  constants) from the XML file created by gencase. Then the methods “Load-
  CaseParticles()” loads the information of the particles from the read input XML,
  the method “ConfigConstants()” calculates all the constants which are required
  for the calculation of the kernels and the method “ConfigDomain()” configures
  the current domain for division.

• Then come methods to initialize the the variables, to update maximum values
  regarding memory, cells used for neighbor list and the particles

• After the initialization, a new method was implemented which was used to set
  initial conditions for the particles of the domain. In this method boundary con-
  ditions are brought about in a spatial manner. At the beginning of the method,
  the spatial limits are read and boundary conditions are brought about in terms of
  fixed values for the primary variable of interest for all particles which are spatially
  located at the boundaries.
• The particle memory data is then printed for the user and the initial state of particles and all the variables involved are stored for the initial output file.

Since DualSPHysics is a free-surface SPH solver, there are situations in impact and splashing problems where particles leave the simulation domain. This also reduces the number of neighbors for each particle which is still in the domain and consequently the accuracy of the SPH summation decreases. To limit this, there is a variable which keeps track of the number of particles excluded from the domain. When this number reaches a predetermined percentage of total particles, the simulation is terminated. This variable is one of the control variables which determines how long the simulation loop runs. When this termination condition is not reached, the simulation runs till the predetermined total simulation time.

The main loop involves three steps which are run repeatedly till the end of the simulation of till the particle exclusion limit is reached. These steps are, calculating the time step, Recalculating the neighbors and saving the data. The computation of the time step is the main step where the core of the SPH solver is implemented which is the calculation of the interaction of particles. This uses the domain division done in the initialization step and for each particle in a cell, neighbors are searched in the same and in the immediately neighboring cells. Then interaction is calculated by the means of the discretized SPH equations explained before. The result of the summation of the forces is the acceleration of the particle, or the rate of change of velocity in the Cauchy momentum equation. The Cauchy momentum equation described before reads

\[
\frac{dv^\alpha_i}{dt} = \sum_j m_j \left( \frac{\sigma^\alpha_j}{\delta_i^2} + \frac{\sigma^\alpha_i}{\delta_j^2} + \Pi_{ij} \delta^{\alpha\beta} + R^{\alpha\beta}_{ij} f^n \right)
\]  

(3.42)

Where \( \Pi_{ij} \) and \( R^{\alpha\beta}_{ij} f^n \) are respectively the artificial viscosity and tensile instability correction, when applied to fluids where the the main volumetric stress is pressure and the main deviatoric part is the due to the viscosity of the fluid, the above equation is modified for fluid flow by replacing \( \sigma \) with \( p \) and by using the physical viscosity formulation in place of artificial viscosity

\[
\frac{dv^\alpha_i}{dt} = \sum_j m_j \left( \frac{P_i}{\delta_i^2} + \frac{P_j}{\delta_j^2} + \Pi_{ij} \delta^{\alpha\beta} + R^{\alpha\beta}_{ij} f^n \right)
\]  

(3.43)

where the viscosity term is given by eq 3.29.

The above two equations use the momentum conserving discretization for the gradient of pressure (or stresses). Two expressions are available for the viscosity, the artificial viscosity explained previously and the Laminar viscosity with sub-particle-scale turbulence from Dalrymple and Rogers (2006). The rate of change of velocity obtained from the sum of the pressure forces and the viscous dissipation of momentum is then
integrated in time with one of the two available time integration schemes to get the acceleration of the particles. This acceleration is then used along with simple mechanics to find the new velocity and the new position of the particles.

3.7 Implementation

As mentioned in chapter 2, the DualSPHysics program was taken as-is and the simple test case was investigated first to recreate turbulent eddies between the armor stones. To accomplish this, the test case of a dambreak with an object in front was taken and was run with one single armor stones in fig. 1.2 simplified as a square block (the object). This line of investigation was not continued because the DualSPHysics version which was used did not have periodic boundary conditions implemented. This implied that the simplified armor stone could not be simulated as completely submerged in turbulent flow.

The focus was then shifted to implement a diffusion equation since the groundwater pressure changes and the processes at the interface due to these changes were of greater importance from the point of view of stability of armoring. The pressure changes caused by the turbulent eddies can be included as boundary conditions imposed on the interface at a later stage. The main comparison to the SPH literature for the above mentioned diffusion problem are the results from Cleary (1998) for heat conduction in solids. To modify the free surface SPH code to be able to solve a diffusion equation in solids, many changes were made to DualSPHysics. First and foremost, the velocity of the particles were initialized to and maintained at zero for the entire simulation time. This mimics a solid body with the density of water. To achieve this the positions and the velocities of the particles were not updated in the time integration. Next, new variables required to model the diffusion equation were introduced, there are mainly the conductivity term, the storage term and the variable to store the value of pressure in terms of meters of water column. The second order derivative for the diffusion equation was implemented preliminarily in the same method which contained the laminar viscosity since the implementation for both the laminar viscosity which has the divergence of the shear stresses, has a similar form to the diffusion term which has the divergence of the gradient of the diffused quantity. The following equation was implemented

$$\frac{\partial h}{\partial t} = \nabla \cdot (\nabla ( - k \nabla h )) = \sum_{j=1}^{n} \frac{m_j}{\rho_i \rho_j} \frac{2 k_i k_j}{k_i + k_j} (h_i - h_j) \frac{r_{ij}}{|r_{ij}|^2} \nabla_i W_{ij}$$  (3.44)

in the above equation the fraction

$$\frac{2k_i k_j}{k_i + k_j}$$

ensures that heterogeneities in conductivity are also accounted for in the calculation of the temperature. The same equation can be used for groundwater flow with the
hydrostatic head $h$ instead of the temperature $T$. The units for the conductivity and storage terms were chosen appropriately.

The boundary conditions were set for the outer most particles of the domain to reflect Dirichlet boundary conditions and were held constant during the time of simulation by resetting the value of the primary variable to the prescribed value at the boundary. The models were set up in two distinct ways. The first model was a square domain with a locally low value for the primary variable at the center of the domain. The simulation was started and the development was observed in time and space. The simulation was stopped before the drawdown reached the boundaries. The snapshots of the initial distribution and the distribution of a diffused quantity ”$x$” is given in the figures 3.4 and 3.5.

Figure 3.4: initial condition of a primitive test case
Next a square model domain was created with and simulations were run varying only one parameter. The conductivity was chosen as the parameter and its value was lowered twice by one order of magnitude from $10^{-4} m/s$ to $10^{-6} m/s$. The simulation was set up with a locally high value for the hydrostatic head $h$ in the middle. The spatial distribution of $h$ in the model domain was observed with time for all three conductivity values and snapshots are given in figures 3.7 and 3.8. From figures it is evident that the conductivity plays an important role in how quickly a locally high value of hydrostatic head (which can also be expressed in terms of pressure) is dissipated into the domain to reach steady state. The less conductive the material is, the slower the front propagates and hence reaching steady state is also much slower. This delay is due to the time taken for water to flow out from the region of high pressure to the region of lower pressure. The low conductivity hinders the flow process keeps the pressure locally high for a longer time.
This is mathematically explained by Darcy’s law for two conductivities $k_1$ and $k_2$. Consider discharges $Q_1$ and $Q_2$ through a unit cross-section for the two cases having the same hydraulic gradient.

\[
Q_1 = -k_1 \nabla h \tag{3.45}
\]

\[
Q_2 = -k_2 \nabla h \tag{3.46}
\]
if

\[ k_1 = 10^{-4} \text{m/s} \]
\[ k_2 = 10^{-6} \text{m/s} = 10^{-2}k_1 \]

then taking the ratio of the equations 3.45 and 3.46

\[ Q_2 = 0.01Q_1 \]

implying that a reduction of the conductivity by two orders of magnitude, reduces the discharge to only one percent of the original discharge and this explains the significant delays in attaining steady state in low conductive soils. The third test case was also a square domain, but unlike the previous model, the initial distribution was done using a Gaussian function instead of a square function. Although with this form of initial condition, the evolution in time should always remain as a Gaussian bell curve, there is a peculiar asymmetry observed in later time. After some discussions this was accounted to possible lack of precision. In the discussion it was suggested that using relative measures with the origin at the center of the domain would theoretically help solve the asymmetry problem occurring at later times. The results are given below, the graphs are plotted along the entire cross section in the x direction at the middle of the domain.

Figure 3.9: Gaussian distribution as initial condition
One test was performed reproducing the conditions used by Cleary (1998). Here a rectangular domain was used and two halves of the domain along its longest axis were held at two different values for \( h \). With this initial condition, the simulation was started and a snapshot was taken after a simulation time of ten seconds. The temperature along the middle of the domain running throughout the x direction was plotted and was compared to the graph reported in literature. The graph of the ten-second snapshot show good agreement with the graph reported in literature. The model parameters used are summarized in table 3.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model Dimension</td>
<td>(1mx0.2m)</td>
</tr>
<tr>
<td>( c_p = k )</td>
<td>1</td>
</tr>
<tr>
<td>Density</td>
<td>1000 kg/m(^3)</td>
</tr>
<tr>
<td>Kernel</td>
<td>Cubic Spline</td>
</tr>
<tr>
<td>Time integration</td>
<td>Verlet</td>
</tr>
</tbody>
</table>

Table 3.1: Parameter value taken from Cleary (1998)
Next, a model of the square domain described in this section was modified to implement a sinusoidal variation of the $h$ in the center of the domain. This was implemented in a straightforward fashion with a regular sine function by choosing several periods. However, there were no concrete tests run to assess the model accuracy since the aim was to implement a preliminary transient boundary condition and to perform a sanity check whether the idea of implementation actually produced results expected from such a system. Hence, no results have been documented here or elaborated.

These models form the first steps towards a coupled, deformable porous media model to simulate the phenomena happening at the interface of the river and the river bed. In the next steps, the disturbances which have been simulated at the center of the
domain, will have to be moved to the upper boundary of the domain. This will replicate the pressure variation due to water level at the interface. This disturbance is then propagated into the ground in a diffusive manner. Because of this there are two possible responses, one is the instantaneous response, which can be approximated as an undrained response in low permeable soils. For this case, the high pore water pressure will prevail while the overburden load on the soil will be removed. The second response is the long-term response, here the soil will additionally show some drainage at the interface due to the reversal of gradient.

These changes in load also cause plastic strains in the soil, the next step after linear elasticity in SPH is to implement plastic strains, failure criterion and a flow rule along the lines of Bui. Once the failure has occurred, the post failure state is also crucial to determine material transport along the bed and to determine the fate of the armoring. To simulate the post-failure processes, the turbulence occurring in waterways has to be simulated with a turbulence model. Turbulence modeling in SPH has already been investigated by many researchers. The next steps is to couple these two models with interface conditions along the lines of Lenaerts.
Chapter 4

Conclusion

In the current research, the processes of interest in waterways from the point of view of bed stability and safety of armoring to erosion have been listed and elaborated. Then a relation has been brought between the various aspects of modelling phenomena at a continuum scale to the processes of interest. These include,

- the definition of the soil as a porous material
- definition of a REV
- mechanical aspects of the soil and soil strength with theory of elasticity and constitutive modelling
- flow through soil at a continuum scale with Darcy’s law
- the influence of the flow on the mechanical response of the soil with the theory of linear poroelasticity
- justification of the use of SPH as a numerical method
- previous work along similar lines in SPH
- available SPH software
- implementation details in DualSPHysics

While referring to the previous work done in SPH for similar problems, the heat conduction problems have been mentioned repeatedly since the literature for heat conduction is quite abundant and that gives a good starting point of implementing equations in SPH. It is also valid since the mathematical theory of linear poroelasticity itself draws parallels between mechanical response of porous materials due to fluid flow through them to the mechanical response of solid materials due to flow of heat through them. Due to this similarity, the SPH implementation of heat conduction problems can be directly adapted and used for problems of flow in poroelastic media.
The current research was aimed to lay a foundation for using SPH at the continuum level applying the well established models of the theory of porous media and elasticity is porous media. Many further steps have to be accomplished to run a test case of the fully coupled model. The most crucial of these is the interface processes between the soil and the flow in the waterway. Although an example from literature has been mentioned in the previous chapter illustrating a possible way this can be done, discussions revealed that its implementation seems to be far from trivial. This was due to the main challenge being converting the diffused pressure into a mass flow at the interface and then the continuous discretization of this mass flow into fluid SPH particles at this interface. This has to be done to obey conservation laws and discussions revealed uncertainty as to where to define the interface where particles can be created and added into the free flow domain without affecting the flow characteristic of the free flow. A suggestion was also made that the particles may be created just inside the porous medium, just below the interface and then added to the free flow region.

Along with this comes the coupling within the soil region itself, between the soil and the groundwater especially when the pore-water pressure is so high that the model experiences irreversible plastic strain. In this region, the model deviates from it’s similarity to thermoelasticity as once the plastic strains occur, the model can be either contractant, dilatant or neutral. This, due to the granular nature of the medium. There has been some examples in literature where the soil has been assumed to be incompressible in the plastic regime, thereby eliminating dilatancy, however, this cannot be taken as a general rule and its validity is case-specific.

Although the physics behind the phenomenon of rapid drawdown in waterways quite complex, the REV scale approach of modelling with the theory of porous media, combined with the application of the meshfree method SPH as the numerical tool gives a wide range of flexibility for modeling. As previously shown in this work, solids can be simulated by setting conditions for particles in the entire domain, and when movement is concerned, SPH is inherently not prone to problems during large deformations as mesh-based methods are. This advantage, together with utilizing previous work done on implementing plastic flow rule and stress correction in SPH gives a strong base on which to extend the diffusion model investigated in this work first to linear elasticity and then to the plastic region of strains, with which the field problem of rapid drawdown can be simulated.
Abbreviations

BAW Bundesanstalt für Wasserbau. 2

CFL Courant-Friedrichs-Lewy. 34

CPU Central Processing Unit. 33, 34

CUDA Compute Unified Device Architecture. 33

DEM Discrete Element Method. 22

Fortran previously FORTRAN, Formula Translating system. 33, 34

GBB Merkblatt Grundlagen zur Bemessung von Böschungs- und Sohlsicherungen an Binnenwasserstraßen. 4

GPU Graphics Processing Unit. 33, 34

GPUSPH a pure GPU implementation of SPH. 33

ISPH Incompressible SPH. 33

MAK Merkblatt Anwendung von Kornfiltran an Bundeswasserstraßen. 4, 20

MAR Merkblatt Anwendung von Regelbauweisen für Böschungs- und Sohlsicherungen an Binnenwasserstraßen. 4

MMB Merkblatt Materialtransport im Boden. 20

PFEM Particle Finite Element Method. 22

REV Representative Elementary Volume. 9–11, 46

SPH Smoothed Particle Hydrodynamics. i, iii, 8, 12, 18, 22–28, 31–37, 44, 46

XML Extensible Markup Language. 34, 35
## Symbols

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Unit</th>
<th>Description</th>
<th>Page(s)</th>
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<td>$\alpha$</td>
<td>$-$</td>
<td>inverse of the storage coefficient</td>
<td>18, 19</td>
</tr>
<tr>
<td>Biot’s modulus</td>
<td>$M$</td>
<td>$-$</td>
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<td>body forces</td>
<td>$f_b$</td>
<td>$N$</td>
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<td>$m/s$</td>
<td>mathematically, the discharge through any soil per unit head gradient</td>
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<tr>
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<td>$h$</td>
<td>$m$</td>
<td>pressure at a point in terms of water column</td>
<td>11, 12</td>
</tr>
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<td>identity matrix</td>
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<td>$m^2$</td>
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<tr>
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<td>$p$</td>
<td>$Pa$</td>
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Bibliography


